CONE CUBIC CONFIGURATIONS OF A RULED SURFACE*

BY

A. F. CARPENTER

I. Introduction

In a series of papers presented to the San Francisco Section of the American Mathematical Society† during the year 1923, the author has discussed numerous characteristics of certain point and line figurations connected with each line element of the general ruled surface. The definitions and theorems of these papers have been built up by the method of the projective differential geometry of Wilczynski from a few fundamental and well known projective properties of ruled surfaces, chief among which are such notions as the flecnode curve and flecnode surface, the complex curve, the osculating quadric and osculating linear complex.

Before proceeding with the present discussion it will be advisable to restate briefly certain of the definitions and theorems involved.

The flecnode curve C_F of a ruled surface S cuts each line element g of S in two points. The planes osculating C_F at two such points intersect in a line h. For each such line there is determined a second line h', the polar reciprocal of h with respect to the linear complex L osculating S along g. The three lines g, h, h' are in general non-intersecting and hence determine a quadric Q_1. The complete intersection of Q_1 and the osculating quadric Q is made up of the element g of S and a space cubic. This curve $C_{F'}$ we call the *primary flecnode cubic.*

To each osculating plane of $C_{F'}$ there corresponds by means of L a point in that plane. The locus of these points for all the osculating planes of $C_{F'}$ is a second space cubic. This curve $C_{F''}$ we call the *secondary flecnode cubic.*

By making use of the complex curve rather than the flecnode curve we arrive at two other cubics called respectively the *primary* and *secondary complex cubics.*

Each space cubic determines a linear complex. The complex L_1 determined by $C_{F'}$ we call the *first associated linear complex.* It develops that $C_{F'}$ *and* $C_{F''}$ *determine the same linear complex.* The linear congruence Γ_1 common to L and L_1 is called the *first associated linear congruence.*

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In a similar manner we obtain from the primary and secondary complex cubics a *second associated linear complex* and a *second associated linear congruence*.

As system of reference in the projective differential geometry of ruled surfaces there is frequently used the tetrahedron $P_y P_z P_\varrho P_\sigma$. Of the four vertices of this tetrahedron P_y and P_z are the flecnode points of $l_{yz} \equiv g$, and P_ϱ, P_σ are two points, one on each of the flecnode tangents $l_{y\varrho}$, $l_{z\sigma}$, drawn to S at P_y, P_z respectively. Unit point in this system is so chosen that the coördinates of the four vertices are P_y, (1, 0, 0, 0); P_z, (0, 1, 0, 0); P_ϱ, (0, 0, 1, 0); P_σ, (0, 0, 0, 1). Of the six edges of the tetrahedron four, namely l_{yz}, $l_{\varrho\sigma}$, $l_{y\sigma}$, $l_{z\varrho}$, belong to L. It follows that the planes which correspond to the points of any one of these lines by means of L constitute a pencil of planes on that line as an axis. There are thus determined four axial pencils.

If, between the points of l_{yz} and $l_{z\varrho}$, there is set up a one-to-one projective correspondence in which to the general point $(\alpha, \beta, 0, 0)$ of l_{yz} there corresponds the point $(0, \alpha, \beta, 0)$ of $l_{z\varrho}$, then there is likewise set up a one-to-one projective correspondence between the planes of the two pencils on these two lines. Since the axes of these two projective pencils intersect, the lines of intersection of corresponding pairs of planes must have for their locus a quadric cone K_1 whose vertex is at P_z. By making use of the pairs of lines $(l_{z\varrho}, l_{\varrho\sigma})$, $(l_{yz}, l_{y\sigma})$, $(l_{y\sigma}, l_{\varrho\sigma})$, in a similar way we define three other cones K_2, K_3, K_4. These four quadric cones we call the *complex cones associated with* g.

Since the pair of lines $(l_{yz}, l_{\varrho\sigma})$ do not intersect, the two pencils of planes on these lines determine a non-developable quadric. It proves to be in fact the osculating quadric, Q. The other pair of non-intersecting lines $(l_{z\varrho}, l_{y\sigma})$ also determines a quadric, Q'; Q and Q' we call the *complex quadrics.* The equations of the four complex cones and the two complex quadrics, in this system of coördinates, are

$$(1)\quad \begin{aligned} &(K_1)\ p_{12}^2 x_1 x_3 + p_{21}^2 x_4^2 = 0, \quad (K_2)\ p_{12}^2 x_1^2 + p_{21}^2 x_2 x_4 = 0, \quad (K_3)\ p_{12}^2 x_3^2 + p_{21}^2 x_2 x_4 = 0,\\ &(K_4)\ p_{12}^2 x_1 x_3 + p_{21}^2 x_2^2 = 0, \quad (Q)\quad x_1 x_4 - \ x_2 x_3 = 0, \quad (Q')\quad x_1 x_2 - \ x_3 x_4 = 0, \end{aligned}$$

where p_{12}, p_{21} are two of the coefficients of the system of differential equations defining the ruled surface S.

The four complex cones K_1, K_2, K_3, K_4 can be paired in six ways. For each of the four pairs $(K_1 K_2)$, $(K_3 K_4)$, $(K_1 K_3)$, $(K_2 K_4)$, the complete intersection is composed of a straight line and a space cubic. These four curves C_1, C_2, C_3, C_4 we call the *primary cone cubics associated with* g. For each of these cubics we indicate below its equations in parametric form, the two cones upon which it lies and the line which completes their intersection:

$$
(2)\quad
\begin{array}{lllllll}
(C_1) & x_1 = -p_{21}^2 t, & x_2 = -p_{21}^2, & x_3 = \;\; p_{12}^2 t^3, & x_4 = \;\; p_{12}^2 t^2; & (K_1K_2), & l_{z\varrho}; \\
(C_2) & x_1 = \;\; p_{12}^2 t^3, & x_2 = \;\; p_{12}^2 t^2, & x_3 = -p_{21}^2 t, & x_4 = -p_{21}^2; & (K_3K_4), & l_{y\sigma}; \\
(C_3) & x_1 = \;\; p_{12}^2 t^3, & x_2 = -p_{21}^2, & x_3 = -p_{21}^2 t, & x_4 = \;\; p_{12}^2 t^2; & (K_1K_3), & l_{yz}; \\
(C_4) & x_1 = -p_{21}^2 t, & x_2 = \;\; p_{12}^2 t^2, & x_3 = \;\; p_{12}^2 t^3, & x_4 = -p_{21}^2; & (K_2K_4), & l_{\varrho\sigma}.
\end{array}
$$

By making use of their osculating planes and the points corresponding to them by means of L, we obtain, from the four cubics $C_1, \cdots, C_4$, four new cubics $C_1', \cdots, C_4'$, just as the secondary flecnode cubic is obtained from the primary flecnode cubic. These four curves $C_1', \cdots, C_4'$ we call the *secondary cone cubics associated with* g. Their equations in parametric form are

$$
(3)\quad
\begin{array}{lllll}
(C_1') & x_1 = p_{21}^3, & x_2 = 3p_{12}p_{21}^2 t, & x_3 = 3p_{12}^2 p_{21} t^2, & x_4 = p_{12}^3 t^3; \\
(C_2') & x_1 = 3p_{12}^2 p_{21} t^2, & x_2 = p_{12}^3 t^3, & x_3 = p_{21}^3, & x_4 = 3p_{12}p_{21}^2 t; \\
(C_3') & x_1 = 3p_{12}^2 p_{21} t^2, & x_2 = -3p_{12}p_{21}^2 t, & x_3 = p_{21}^3, & x_4 = -p_{12}^3 t^3; \\
(C_4') & x_1 = p_{21}^3, & x_2 = -p_{12}^3 t^3, & x_3 = 3p_{12}^2 p_{21} t^2, & x_4 = -3p_{12}p_{21}^2 t.
\end{array}
$$

The parametric equations of the primary and secondary flecnode cubics, and the equation, in line coördinates, of the linear complex L_1 which they determine, are respectively

$$
(4)\quad
\begin{array}{lll}
(C_F) & x_1 = 2t(p_{12}^2 q_{21} t^2 - p_{21}^2 q_{12}), & x_2 = 2(p_{12}^2 q_{21} t^2 - p_{21}^2 q_{12}), \\
 & x_3 = p_{12} p_{21} t\,(p_{12} t^2 - p_{21}), & x_4 = p_{12} p_{21}\,(p_{12} t^2 - p_{21}); \\
(C_{F'}) & x_1 = 2p_{21}(3p_{12}^2 q_{21} t^2 + p_{21}^2 q_{12}), & x_2 = 2p_{12} t\,(p_{12}^2 q_{21} t^2 + 3p_{21}^2 q_{12}), \\
 & x_3 = p_{12} p_{21}^2\,(3p_{12} t^2 + p_{21}), & x_4 = p_{12}^2 p_{21} t\,(p_{12} t^2 + 3p_{21});
\end{array}
$$

$$
(L_1)\quad 2p_{12}p_{21}\omega_{12} - (p_{12}q_{21} + 3p_{21}q_{12})\omega_{14} + (3p_{12}q_{21} + p_{21}q_{12})\omega_{23} + 8q_{12}q_{21}\omega_{34} = 0;
$$

where q_{12}, q_{21}, are another pair of coefficients of the system of differential equations defining S.

The equation of the quadric Q_1 is

$$
(Q_1)\quad p_{12}^2 p_{21} x_1 x_3 - p_{12} p_{21}^2 x_2 x_4 - 2p_{12}^2 q_{21} x_3^2 + 2p_{21}^2 q_{12} x_4^2 = 0.
$$

If we identify the parameters in the equations of the ten cubics listed above we thereby set up a point correspondence between these curves. But this correspondence is not arbitrary. That between C_F and $C_{F''}$ is indeed the one resulting from the definition of $C_{F''}$. The projective transformation*

$$(5_1)\qquad \begin{aligned} \bar{x}_1 &= -p_{21}x_1 + 2q_{21}x_3, & \bar{x}_2 &= -p_{21}x_2 + 2q_{21}x_4,\\ \bar{x}_3 &= p_{12}x_1 - 2q_{12}x_3, & \bar{x}_4 &= p_{12}x_2 - 2q_{12}x_4 \end{aligned}$$

carries $C_{F'}$ into C_1 and $C_{F''}$ into C_1'. An examination of equations (2) shows that the four curves $C_1, \cdots, C_4$ are projectively equivalent, the projectivities which carry C_1 into C_2, C_3, C_4 being respectively

$$(5_2)\qquad \bar{x}_1 = x_3,\quad \bar{x}_2 = x_4,\quad \bar{x}_3 = x_1,\quad \bar{x}_4 = x_2;$$

$$(5_3)\qquad \bar{x}_1 = x_3,\quad \bar{x}_2 = x_2,\quad \bar{x}_3 = x_1,\quad \bar{x}_4 = x_4;$$

$$(5_4)\qquad \bar{x}_1 = x_1,\quad \bar{x}_2 = x_4,\quad \bar{x}_3 = x_3,\quad \bar{x}_4 = x_2.$$

Similarly for the curves $C_1', \cdots, C_4'$, but with a different set of projectivities. The point correspondence between the five curves $C_{F'}$, C_1, C_2, C_3, C_4, as well as that between the five curves $C_{F''}$, C_1', C_2', C_3', C_4', is therefore projective in nature, while that between the pairs C_j, C_j' $(j = 1, \cdots, 4)$, is exactly that determined by the complex L between $C_{F'}$ and $C_{F''}$.

From the general theory† a fundamental system of simultaneous solutions $y_k(x)$, $z_k(x)$ $(k = 1, \cdots, 4)$ of the system of equations defining S determines two directrix curves C_y, C_z of S. Corresponding points P_y, P_z of these curves have the homogeneous coördinates (y_1, y_2, y_3, y_4), (z_1, z_2, z_3, z_4) and lie upon the same generator g of S. For our choice of a coördinate system, as we have seen, $y_1 = 1$, $y_2 = y_3 = y_4 = 0$; $z_2 = 1$, $z_1 = z_3 = z_4 = 0$; while $\varrho_3 = 1$, $\varrho_1 = \varrho_2 = \varrho_4 = 0$; $\sigma_4 = 1$, $\sigma_1 = \sigma_2 = \sigma_3 = 0$; where

$$(6)\qquad \varrho = 2\frac{dy}{dx} + p_{12}z, \qquad \sigma = 2\frac{dz}{dx} + p_{21}y.$$

For the general point P_x; (a, b, c, d), the coördinates are given by the expression

$$x_k = a y_k + b z_k + c \varrho_k + d\sigma_k \qquad (k = 1, \cdots, 4),$$

the subscript k ordinarily being omitted.

* p_{12}, p_{21}, q_{12}, q_{21} are functions of a parameter x which is constant so long as we are considering a single line element g of S. Hence this transformation is projective for all configurations associated with g.

† *Projective Differential Geometry of Curves and Ruled Surfaces*, Wilczynski; B. G. Teubner, 1906, pp. 129, 130. Hereafter referred to as Proj. Dif. Geom.

It is the purpose of this paper to characterize with reasonable completeness the configurations formed by the ten curves $C_{F'}, C_{F''}; C_1, \cdots, C_4; C_1', \cdots, C_4'$, to define and discuss two additional sets of cubics, and to suggest further problems for investigation.

II. The first and second cone rays

There are four pencils of quadrics determined by the pairs of cones $(K_1 K_2)$, $(K_3 K_4)$, $(K_1 K_3)$, $(K_2 K_4)$, namely

$$(7)\quad \begin{aligned} &h(p_{12}^2 x_1 x_3 + p_{21}^2 x_4^2) + k(p_{12}^2 x_1^2 + p_{21}^2 x_2 x_4) = 0,\\ &h(p_{12}^2 x_3^2 + p_{21}^2 x_2 x_4) + k(p_{12}^2 x_1 x_3 + p_{21}^2 x_2^2) = 0,\\ &h(p_{12}^2 x_1 x_3 + p_{21}^2 x_4^2) + k(p_{12}^2 x_3^2 + p_{21}^2 x_2 x_4) = 0,\\ &h(p_{12}^2 x_1^2 + p_{21}^2 x_2 x_4) + k(p_{12}^2 x_1 x_3 + p_{21}^2 x_2^2) = 0. \end{aligned}$$

By using the same parameters in each of the four equations (7) we have set up an arbitrary correspondence between the quadrics of these four pencils. We do this as a matter of convenience.

C_1 lies upon each of the quadrics of (7_1). Since C_1 also lies upon Q each of these quadrics (7_1) must have in common with Q one, and only one, straight line. It proves to be the line joining the point $hy-k\varrho$ on $l_{y\varrho}$ to the point $hz-k\sigma$ on $l_{z\sigma}$. As we pass from surface to surface of the pencil (7_1), this line coincides successively with all the rulings of Q.

C_2 lies upon each of the quadrics of (7_2). Since C_2 also lies upon Q, each of these quadrics (7_2) must have in common with Q one, and only one, straight line. It also proves to be the line determined by the points $hy-k\varrho$ and $hz-k\sigma$, so that corresponding quadrics of the pencils (7_1) and (7_2) cut Q in the same line.

The cubics C_3, C_4 are seen by inspection to lie upon the quadric Q'. But C_3 lies upon each quadric of the pencil (7_3) and C_4 upon each quadric of the pencil (7_4). Each quadric of (7_3), as also each of (7_4), must therefore have one, and only one, straight line in common with Q'. This line is seen to be that which is determined by the two points $ky-h\varrho$, $hz-k\sigma$ on $l_{y\varrho}$, $l_{z\sigma}$, respectively. Corresponding pairs of quadrics of the two pencils (7_3), (7_4) have this line in common with Q'. As we pass from surface to surface of (7_3) or (7_4), this line coincides successively with all of the rulings of Q'.

If we write for the general point on the first of these lines the expression

$$f(hy-k\varrho)+g(hz-k\sigma),$$

and, for the general point on the second, the expression

$$m(ky-h\varrho)+n(hz-k\sigma),$$

then it follows that these two points will coincide if and only if

$$\frac{h}{k}=\frac{k}{h}, \text{ that is, if } h=\pm k.$$

One may thus in two ways choose a set of four quadrics, one from each pencil, such that the four will have a line in common and this line will at the same time lie upon both the complex quadrics. The points in which these two lines cut the flecnode tangents $l_{y\varrho}$, $l_{z\sigma}$ will engage our attention again. They are given by the expressions

$$(8) \qquad \alpha=y-\varrho, \quad \beta=z-\sigma; \qquad \gamma=y+\varrho, \quad \delta=z+\sigma.$$

We note that P_α, P_γ *are harmonic conjugates with respect to* P_y, P_ϱ, *as are also* P_β, P_δ, *with respect to* P_z, P_σ.

The cones K_1 and K_2 have C_1 and $l_{z\varrho}$ as their complete intersection. The line $l_{z\varrho}$ is tangent to K_3 at P_z but is not an element of K_3. If we substitute from (C_1) of (2) into (K_3) of (1) we find for t the six values

$$0, \quad 0, \quad p_{21}/p_{12}, \quad -p_{21}/p_{12}, \quad p_{21}i/p_{12}, \quad -p_{21}i/p_{12}.$$

The first two of these correspond to the point P_z and the remaining four, by means of (C_1), give respectively the four points P_φ, P_ψ, P_η, P_ζ, where

$$(9) \qquad \begin{aligned} \varphi &= p_{21}\ y+p_{12}z-p_{21}\ \varrho-p_{12}\sigma = p_{21}\ \alpha+p_{12}\beta, \\ \psi &= p_{21}\ y-p_{12}z-p_{21}\ \varrho+p_{12}\sigma = p_{21}\ \alpha-p_{12}\beta, \\ \eta &= p_{21}iy+p_{12}z+p_{21}i\varrho+p_{12}\sigma = p_{21}i\gamma+p_{12}\delta, \\ \zeta &= p_{21}iy-p_{12}z+p_{21}i\varrho-p_{12}\sigma = p_{21}i\gamma-p_{12}\delta. \end{aligned}$$

From (K_4) of (1), the point P_z is not on K_4, but the four points P_φ, P_ψ, P_η, P_ζ are. Moreover, (9) shows that these four points are also on Q and Q'. Two of them, P_φ, P_ψ, are in fact harmonic conjugates with respect to P_α, P_β, and the other two, P_η, P_ζ, harmonic conjugates with respect to P_γ, P_δ. These four points P_φ, P_ψ, P_η, P_ζ, are the only ones common to all four cones $K_1, \cdots, K_4$. Since they may be thought of as determining the two lines $l_{\alpha\beta}$, $l_{\gamma\delta}$, we shall speak of these two lines as the *first and second cone rays associated with* g.

The four points P_φ, P_ψ, P_η, P_ζ have already been shown to lie on C_1. By introducing the values $t = \pm p_{21}/p_{12}$, $\pm p_{21} i/p_{12}$ into (C_2), (C_3), (C_4) of (2) *it results that all four curves pass through these four points and that in the point correspondence existing between the four curves, these four points are self-corresponding.* It should be noted that P_φ, thought of as being on C_1, C_2, corresponds to P_ψ, thought of as being on C_3, C_4, and conversely, while P_η counts for a complete set of four corresponding points, as does also P_ζ.

All four points cannot be real at the same time, the two on the second cone ray, $l_{\gamma\delta}$, being imaginary if those on the first cone ray, $l_{\alpha\beta}$, are real, and conversely. It may happen that none of the four are real. We shall speak of these four points as the *focal points of the primary cone cubics.*

If (f, g, h, k) be any point, then the three points whose coördinates are (h, k, f, g), (h, g, f, k), (f, k, h, g), as well as the three points whose coördinates are (h, k, f, g), $(h, -g, f, -k)$, $(f, -k, h, -g)$, determine a plane on which the first point lies. It follows from equations (5) that if P_1, P_2, P_3, P_4 are a set of four corresponding points, one on each of the curves C_1, C_2, C_3, C_4, then P_1, P_2, P_3, P_4 are coplanar. Likewise it follows that if P_1', P_2', P_3', P_4' are a set of four corresponding points, one on each of the curves C_1', C_2', C_3', C_4', then P_1', P_2', P_3', P_4' are coplanar. The equations of the planes p and p' containing these two sets of points are easily found. For the first set we find the equation

$$x_1 + x_3 - t(x_2 + x_4) = 0, \tag{10}$$

and for the second set, the equation

$$p_{12}\varkappa t(x_1 - x_3) - p_{21}\lambda(x_2 - x_4) = 0, \tag{11}$$

where

$$\varkappa = p_{12}^2 t^2 - 3p_{21}^2, \qquad \lambda = 3p_{12}^2 t^2 - p_{21}^2. \tag{12}$$

The points $\alpha = y - \varrho$, $\beta = z - \sigma$ are on the first of these two planes and the points $\gamma = y + \varrho$, $\delta = z + \sigma$ are on the second. It follows that equations (10) and (11) define two pencils of planes whose axes are the first and second cone rays, t being the parameter of the pencil in each case. In brief, *the projectivity existing between the points of the four primary (secondary) cone cubics is such that corresponding points lie by fours in the planes of an axial pencil whose axis is the first (second) cone ray.* We note from (4), (10) and (11) that the point on the primary flecnode cubic which corresponds to the set of points $P_1, \cdots, P_4$ lies on their plane, but that the point of the secondary flecnode cubic which corresponds to the set $P_1', \cdots, P_4'$ lies on their plane only if $p_{12} - 2q_{12} = p_{21} - 2q_{21}$.

III. Perspectivities of the cone cubics

Since the cubics C_1 and C_2 lie on Q while the cubics C_3 and C_4 lie on Q', it results that of the four points P_1, P_2, P_3, P_4, in which these four curves cut the plane p, P_1 and P_2 lie on Q and P_3 and P_4 upon Q'. Since $l_{\alpha\beta}$ lies on both Q and Q', the plane p through this line must be a tangent plane to both quadrics. Its points of tangency with Q and Q' are given by the respective expressions

$$(13) \qquad \tau = t\alpha + \beta, \qquad \tau' = -t\alpha + \beta.$$

These points are harmonic conjugates with respect to P_α and P_β. We note that the points P_1, P_2 are not on $l_{\alpha\beta}$. They must therefore lie upon the second line which p has in common with Q, so that *P_1, P_2 and P_τ are collinear.* A similar argument leads to the conclusion that *P_3, P_4 and $P_{\tau'}$ are collinear.* We find further that *P_1, P_3 and P_α are collinear as are also the sets P_2, P_4, P_α; P_1, P_4, P_β; P_2, P_3, P_β.*

The lines P_1P_2 and P_3P_4, lying in plane p, intersect. But P_1P_2 is a ruling of Q and P_3P_4 a ruling of Q'. Their point of intersection must therefore be a point of the intersection of Q and Q'. The complete intersection of these two quadrics consists of the two flecnode tangents $l_{y\varrho}$, $l_{z\sigma}$, and the two cone rays $l_{\alpha\beta}$, $l_{\gamma\delta}$. Moreover the flecnode tangents belong to one regulus and the cone rays to the other regulus, on each quadric. Since P_1P_2 and P_3P_4 intersect $l_{\alpha\beta}$ in distinct points, their point of intersection must be on $l_{\gamma\delta}$. It is indeed the point given by the expression

$$\theta = t\gamma + \delta.$$

To sum up, we find that *the complete quadrangle whose vertices $P_1, \cdots, P_4$ are any set of corresponding primary cone cubic points lies on a plane which is at the same time tangent to both of the complex quadrics Q and Q'. Of its six sides, one, P_1P_2, lies on Q and another, P_3P_4, lies on Q'. Of its diagonal points, two, P_α, P_β, lie upon the first cone ray and are the same for all such quadrangles, while the third lies upon the second cone ray.*

That plane of the pencil (10) which is tangent to Q at the point P_{τ_1}, where $\tau_1 = t_1\alpha + \beta$, is given by the equation

$$(10_1) \qquad x_1 + x_3 - t_1(x_2 + x_4) = 0.$$

The tangent plane to Q' at this same point has for its equation

$$(10_{-1}) \qquad x_1 + x_3 + t_1(x_2 + x_4) = 0.$$

But the plane (10_1) is tangent to Q' at $P_{\tau_{-1}}$, where $\tau_{-1} = -t_1\alpha + \beta$, and the plane (10_{-1}) is tangent to Q at this same point.

Let us think of the planes of the pencil (10) as paired in this way, the two planes of such a pair being indicated by p_1 and p_{-1} and the quadrangles in these planes by $(P_1P_2P_3P_4)$ and $(P_{-1}P_{-2}P_{-3}P_{-4})$. Then it is seen that in either of the two orders

$$(P_1P_2P_3P_4) \sim (P_{-3}P_{-4}P_{-1}P_{-2}) \text{ or } (P_1P_2P_3P_4) \sim (P_{-4}P_{-3}P_{-2}P_{-1}),$$

we can determine a correspondence between these two quadrangles such that corresponding pairs of lines intersect in $l_{\alpha\beta}$. We find indeed that the lines P_1P_3, P_2P_4, $P_{-1}P_{-3}$, $P_{-2}P_{-4}$ intersect in P_α, the lines P_1P_4, P_2P_3, $P_{-1}P_{-4}$, $P_{-2}P_{-3}$ intersect in P_β, the lines P_1P_2, $P_{-3}P_{-4}$, in the point P_τ and the lines P_3P_4, $P_{-1}P_{-2}$, in the point $P_{\tau'}$.

It follows that in two ways these two quadrangles are in perspective from a point. It can be easily verified that the four lines P_1P_{-3}, P_2P_{-4}, P_3P_{-1}, P_4P_{-2} pass through the point P_γ, while the lines P_1P_{-4}, P_2P_{-3}, P_3P_{-2}, P_4P_{-1} pass through the point P_δ.

Without further discussion we remark that if the planes of the pencil on the second cone ray $l_{\gamma\delta}$ are paired as above, so that of the two in each pair, the first, q_1,

$$x_1 - x_3 - t_1(x_2 - x_4) = 0,$$

is tangent to Q at $\theta_1 = t_1\gamma + \delta$, while the second, q_{-1},

$$x_1 - x_3 + t_1(x_2 - x_4) = 0,$$

is tangent to Q' at the same point, then the quadrangle $(P_1P_2P_{-3}P_{-4})$ lies in the first of these planes, the quadrangle $(P_3P_4P_{-1}P_{-2})$ lies in the second, and these two quadrangles are in perspective from the two points P_α, P_β, the four lines P_1P_3, P_2P_4, $P_{-3}P_{-1}$, $P_{-4}P_{-2}$ passing through P_α and the four lines P_1P_4, P_2P_3, $P_{-3}P_{-2}$, $P_{-4}P_{-1}$ through P_β.

To recapitulate: let those points of the first and second cone rays $l_{\alpha\beta}$, $l_{\gamma\delta}$ correspond which lie upon the same line of that regulus of Q to which the flecnode tangents belong. At each of such a pair of corresponding points P_τ, P_θ, construct the tangent planes to Q and Q'. Of these four planes, two, p_1, p_{-1}, are on $l_{\alpha\beta}$, and two, q_1, q_{-1}, are on $l_{\gamma\delta}$. Of the total of $3\cdot 4\cdot 4 = 48$ points of intersection of these four planes with the four primary cone cubics, 32 coincide by eights in the four focal points P_φ, P_ψ, P_η, P_ζ. The remaining 16 points coincide by twos.

Of these eight distinct points four lie in each of the planes p_1, p_{-1}, q_1, q_{-1}, in such a way that no plane contains more than one point from each cone cubic. Consider the points P_α, P_β, and P_γ, P_δ, in which $l_{\alpha\beta}$ and $l_{\gamma\delta}$ are cut by $l_{\alpha\gamma}$ and $l_{\beta\delta}$. The lines joining either P_γ or P_δ to the four cone cubic points in either p_1 or p_{-1} pass through the cone cubic points in the other of these two planes, and the lines joining either P_α or P_β to the four cone cubic points in either q_1 or q_{-1} pass through the four cone cubic points in the other of these two planes.

We have already seen that, for each set of corresponding points P_1, P_2, P_3, P_4, lying in the plane p_1, there exists a set of corresponding secondary cone cubic points P_1', P_2', P_3', P_4', lying in the plane p_1',

$$(11_1)\qquad p_{12}\varkappa_1 t_1(x_1-x_3)-p_{21}\lambda_1(x_2-x_4) = 0.$$

Similarly, for the set of points P_{-1}, P_{-2}, P_{-3}, P_{-4}, lying in the plane p_{-1}, there is a set P_{-1}', P_{-2}', P_{-3}', P_{-4}', lying in the plane p_{-1}',

$$(11_{-1})\qquad p_{12}\varkappa_1 t_1(x_1-x_3)+p_{21}\lambda_1(x_2-x_4) = 0.$$

For these two quadrangles lying in the planes p_1', p_{-1}', it also holds that in either of two orders,

$$(P_1'P_2'P_3'P_4')\sim(P_{-3}'P_{-4}'P_{-1}'P_{-2}') \quad\text{or}\quad (P_1'P_2'P_3'P_4')\sim(P_{-4}'P_{-3}'P_{-2}'P_{-1}'),$$

we can determine a correspondence between them such that corresponding pairs of lines intersect on $l_{\gamma\delta}$. We find that the lines $P_1'P_3'$, $P_2'P_4'$, $P_{-1}'P_{-3}'$, $P_{-2}'P_{-4}'$ intersect in P_γ, the lines $P_1'P_4'$, $P_2'P_3'$, $P_{-1}'P_{-4}'$, $P_{-2}'P_{-3}'$, in P_δ, the lines $P_1'P_2'$, $P_{-3}'P_{-4}'$, in P_ξ, and the lines $P_3'P_4'$, $P_{-1}'P_{-2}'$, in $P_{\xi'}$, where

$$(14)\qquad \begin{aligned} \xi &= p_{21}\nu\gamma+p_{12}\mu t\delta, & \xi' &= p_{21}\nu\gamma-p_{12}\mu t\delta,\\ \nu &= 3p_{12}^2t^2+p_{21}^2, & \mu &= p_{12}^2t^2+3p_{21}^2. \end{aligned}$$

It follows that in two ways these two quadrangles are in perspective from a point. Indeed the lines $P_1'P_{-3}'$, $P_2'P_{-4}'$, $P_3'P_{-1}'$, $P_4'P_{-2}'$ pass through P_α, and the lines $P_1'P_{-4}'$, $P_2'P_{-3}'$, $P_3'P_{-2}'$, $P_4'P_{-1}'$ pass through P_β.

If on the other hand we regroup the points into the sets $(P_1'P_2'P_{-3}'P_{-4}')$, $(P_{-1}'P_{-2}'P_3'P_4')$, we find that these two sets lie in the respective planes

$$p_{12}\mu_1 t_1(x_1+x_3)-p_{21}\nu_1(x_2+x_4) = 0,$$

$$p_{12}\mu_1 t_1(x_1+x_3)+p_{21}\nu_1(x_2+x_4) = 0,$$

on the first cone ray $l_{\alpha\beta}$. Moreover these two quadrangles are in perspective from the two points P_γ, P_δ, the four lines $P_1'P_3'$, $P_2'P_4'$, $P_{-3}'P_{-1}'$, $P_{-4}'P_{-2}'$ passing through P_γ, and the four lines $P_1'P_4'$, $P_2'P_3'$, $P_{-3}'P_{-2}'$, $P_{-4}'P_{-1}'$ passing through P_δ.

The equations of the planes osculating the four primary cone cubics at a set of corresponding points P_1, P_2, P_3, P_4 are, respectively,

$$(15)\quad \begin{aligned} &(\pi_1)\quad 3p_{12}^2 t^2 x_1 - p_{12}^2 t^3 x_2 - p_{21}^2 x_3 + 3p_{21}^2 t\, x_4 = 0,\\ &(\pi_2)\quad p_{21}^2 x_1 - 3p_{21}^2 t\, x_2 - 3p_{12}^2 t^2 x_3 + p_{12}^2 t^3 x_4 = 0,\\ &(\pi_3)\quad p_{21}^2 x_1 + p_{12}^2 t^3 x_2 - 3p_{12}^2 t^2 x_3 - 3p_{21}^2 t\, x_4 = 0,\\ &(\pi_4)\quad 3p_{12}^2 t^2 x_1 + 3p_{21}^2 t\, x_2 - p_{21}^2 x_3 - p_{12}^2 t^3 x_4 = 0. \end{aligned}$$

These four planes pass through a common point P_u on $l_{\gamma\delta}$, where

$$(16)\qquad u = \mu t\gamma + \nu\delta.$$

The equations of the planes osculating the four secondary cone cubics at the points P_1', P_2', P_3', P_4' are, respectively,

$$(17)\quad \begin{aligned} &(\pi_1')\quad p_{12}^3 t^3 x_1 - p_{12}^2 p_{21} t^2 x_2 + p_{12} p_{21}^2 t\, x_3 - p_{21}^3 x_4 = 0,\\ &(\pi_2')\quad p_{12} p_{21}^2 t\, x_1 - p_{21}^3 x_2 + p_{12}^3 t^3 x_3 - p_{12}^2 p_{21} t^2 x_4 = 0,\\ &(\pi_3')\quad p_{12} p_{21}^2 t\, x_1 + p_{12}^2 p_{21} t^2 x_2 + p_{12}^3 t^3 x_3 + p_{21}^3 x_4 = 0,\\ &(\pi_4')\quad p_{12}^3 t^3 x_1 + p_{21}^3 x_2 + p_{12} p_{21}^2 t\, x_3 + p_{12}^2 p_{21} t^2 x_4 = 0. \end{aligned}$$

These four planes pass through a common point P_v on $l_{\alpha\beta}$, where

$$(18)\qquad v = p_{21}\alpha + p_{12} t\beta.$$

The common tangent planes to Q and Q' at the points P_α, P_β are given by the equations

$$(19)\qquad (c)\quad x_2 + x_4 = 0, \qquad (d)\quad x_1 + x_3 = 0.$$

From (17) and (19) we discover that the following sets of planes are collinear:

$$(\pi_1', \pi_3', d), \quad (\pi_2', \pi_4', d), \quad (\pi_1', \pi_4', c), \quad (\pi_2', \pi_3', c).$$

The plane determined by the line of intersection of π_1', π_2' and the line $l_{\alpha\beta}$ has for its equation

$$(20)\qquad p_{12} t(x_1 + x_3) - p_{21}(x_2 + x_4) = 0,$$

and the plane determined by the line of intersection of π_3', π_4' and the line $l_{\alpha\beta}$ has for its equation

$$(21) \qquad p_{12}\, t(x_1 + x_3) + p_{21}(x_2 + x_4) = 0.$$

The four planes (c), (d), (20), (21) are all on $l_{\alpha\beta}$, the second pair being harmonic conjugates with respect to the first pair, and vice versa. We may write the equations of the planes osculating the four secondary cone cubics at the points P_{-1}', P_{-2}', P_{-3}', P_{-4}' by replacing t with $-t$ in (17). These new planes will pass through the point

$$v' = p_{21}\,\alpha - p_{12}\, t\beta,$$

on $l_{\alpha\beta}$, and for this set of planes the following triples are collinear:

$$(\pi_{-1}', \pi_{-3}', d), \quad (\pi_{-2}', \pi_{-4}', d), \quad (\pi_{-1}', \pi_{-4}', c), \quad (\pi_{-2}', \pi_{-3}', c).$$

The plane determined by the line of intersection of π_{-1}', π_{-2}' and $l_{\alpha\beta}$ is precisely that given by (21), while the plane determined by the line of intersection of π_{-3}', π_{-4}' and $l_{\alpha\beta}$ is given by (20). It follows that in either of two orders,

$$(\pi_1'\,\pi_2'\,\pi_3'\,\pi_4') \sim (\pi_{-3}'\,\pi_{-4}'\,\pi_{-1}'\,\pi_{-2}') \quad \text{or} \quad (\pi_1'\,\pi_2'\,\pi_3'\,\pi_4') \sim (\pi_{-4}'\,\pi_{-3}'\,\pi_{-2}'\,\pi_{-1}'),$$

we can determine a correspondence between these two sets of four planes such that corresponding pairs of lines determine planes on the line $l_{\alpha\beta}$. We find in fact that the lines $\pi_1'\pi_3'$, $\pi_2'\pi_4'$, $\pi_{-1}'\pi_{-3}'$, $\pi_{-2}'\pi_{-4}'$ lie on plane d, the lines $\pi_1'\pi_4'$, $\pi_2'\pi_3'$, $\pi_{-1}'\pi_{-4}'$, π_{-2}', π_{-3}' lie on plane c, the lines $\pi_1'\pi_2'$, $\pi_{-3}'\pi_{-4}'$ lie on plane (20), and the lines $\pi_3'\pi_4'$, $\pi_{-1}'\pi_{-2}'$ lie on plane (21).

It results that in either of two ways these two sets of four planes are in perspective from a plane. It can be verified that the four lines $\pi_1'\pi_{-3}'$, $\pi_2'\pi_{-4}'$, $\pi_3'\pi_{-1}'$, $\pi_4'\pi_{-2}'$ are coplanar, as are also the four lines $\pi_1'\pi_{-4}'$, $\pi_2'\pi_{-3}'$, $\pi_3'\pi_{-2}'$, $\pi_4'\pi_{-1}'$, the two planes having for their respective equations

$$(22) \qquad (b) \quad x_1 - x_3 = 0, \qquad (a) \quad x_2 - x_4 = 0.$$

Planes (a), (b) are common tangent planes to Q and Q' at the respective points P_γ, P_δ, on $l_{\gamma\delta}$.

Without further discussion we note that in either of two ways the two sets of planes $(\pi_1'\,\pi_2'\,\pi_{-3}'\,\pi_{-4}')$, $(\pi_3'\,\pi_4'\,\pi_{-1}'\,\pi_{-2}')$ are in perspective from a plane, the four lines $\pi_1'\pi_3'$, $\pi_2'\pi_4'$, $\pi_{-3}'\pi_{-1}'$, $\pi_{-4}'\pi_{-2}'$ lying on the plane (d) and the four lines $\pi_1'\pi_4'$, $\pi_2'\pi_3'$, $\pi_{-3}'\pi_{-2}'$, $\pi_{-4}'\pi_{-1}'$ lying on the plane (c). Similar perspectivities exist between the two sets of planes π_1, π_2, π_3, π_4, and π_{-1}, π_{-2}, π_{-3}, π_{-4}. It is not necessary to dwell further upon this.

We close this part of our discussion by emphasizing the duality which exists between the primary cone cubics, thought of as point loci, and the secondary cone cubics, thought of as the loci of their osculating planes, this duality being of a reciprocal nature.

IV. Other properties of the cone cubics. Allied curves

The perspectivities of the cone cubics are by no means their only interesting properties. Without going into unnecessary detail, we will establish in this section a number of theorems which will serve to illustrate the wealth of material awaiting further investigation.

Each of the primary cone cubics determines a linear complex*. Since $C_1, \cdots, C_4$ are projectively equivalent to the primary flecnode cubic C_F, the corresponding four linear complexes will be projectively equivalent to to the complex L_1. We obtain the equations, in line coördinates, of these four complexes by applying to (L_1) of (4) the transformations of line coördinates which are the consequences of the four transformations (5_1), (5_2), (5_3), (5_4) in point coördinates. We write below, for each of these complexes, its equation in line coördinates together with the point-plane correspondence which it determines. We have

$$(23)\quad \begin{array}{lllll} (L_{11})\ 3\omega_{14}-\omega_{23}=0, & u_1=3x_4, & u_2=-x_3, & u_3=x_2, & u_4=-3x_1; \\ (L_{12})\ \omega_{14}-3\omega_{23}=0, & u_1=x_4, & u_2=-3x_3, & u_3=3x_2, & u_4=-x_1; \\ (L_{13})\ \omega_{12}+3\omega_{34}=0, & u_1=x_2, & u_2=-x_1, & u_3=3x_4, & u_4=-3x_3; \\ (L_{14})\ 3\omega_{12}+\omega_{34}=0, & u_1=3x_2, & u_2=-3x_1, & u_3=x_4, & u_4=-x_3. \end{array}$$

The points $A_1, \cdots, A_4$ which correspond to plane (10) by means of these four linear complexes have for their coördinates, according to (23),

$$(24)\quad \begin{array}{rcccc} & A_1 & A_2 & A_3 & A_4 \\ x_1= & t, & 3t, & 3t, & t, \\ x_2= & 3, & 1, & 3, & 1, \\ x_3= & 3t, & t, & t, & 3t, \\ x_4= & 1, & 3, & 1, & 3. \end{array}$$

The loci $a_1, \cdots, a_4$ of these four points are of course straight lines, the polar reciprocals of $l_{\alpha\beta}$ with respect to the four complexes $L_{11}, \cdots, L_{14}$.

* The four linear complexes determined by the secondary cone cubics are identical with those determined by the primary cone cubics.

From (24) and (2) we find that the quadrangles $P_1P_2P_3P_4$ and $A_1A_2A_3A_4$, both lying in plane (10), are in perspective from the point P; $(t, 1, t, 1)$, in which this plane is cut by $l_{\gamma\delta}$. In brief, *the polar reciprocals of the first cone ray, taken with respect to the linear complexes determined by the primary cone cubics, determine on each plane of the primary pencil associated with g a quadrangle which is in perspective with the quadrangle of the primary cone cubic points of this plane, the locus of the center of perspective being the second cone ray.*

A number of similar theorems may be readily obtained by interchanging cone rays and by making use of the secondary, rather than the primary, cone cubic points. We leave these to be enunciated by the reader.

We have already seen that the equation of the general plane on the second cone ray is

$$x_1 - x_3 - t(x_2 - x_4) = 0. \tag{25}$$

The points $B_1, \cdots, B_4$ which correspond to the plane (25) by means of the four complexes $L_{11}, \cdots, L_{14}$ and whose loci $b_1, \cdots, b_4$ are the polar reciprocals of $l_{\gamma\delta}$, have for their coördinates, by (23),

$$\begin{array}{rrrrr} & B_1 & B_2 & B_3 & B_4 \\ x_1 = & t, & 3t, & 3t, & t, \\ x_2 = & 3, & 1, & 3, & 1, \\ x_3 = & -3t, & -t, & -t, & -3t, \\ x_4 = & -1, & -3, & -1, & -3. \end{array} \tag{26}$$

From (24), (26), and (1) we find that, of the eight lines involved, a_1, a_2, b_1, b_2 lie on Q' and a_3, a_4, b_3, b_4 lie on Q. Moreover the points (a_1a_4), (a_2a_3), (b_1b_4), (b_2b_3) are on the flecnode tangent $l_{y\varrho}$ and the points (a_1a_3), (a_2a_4), (b_1b_3), (b_2b_4) are on the flecnode tangent $l_{z\sigma}$. Summing up these results we find that *the four polar reciprocals of the first* (*second*) *cone ray, taken with respect to the linear complexes determined by the primary cone cubics associated with g, constitute four edges of a tetrahedron whose other two edges are the flecnode tangents, and of these four lines two lie upon each of the complex quadrics.*

The four planes (17) are in general distinct, but when $t = p_{21}/p_{12}$, the first and second coincide in the plane whose equation is

$$x_1 - x_2 + x_3 - x_4 = 0, \tag{27_1}$$

and the third and fourth coincide in the plane whose equation is

$$x_1 + x_2 + x_3 + x_4 = 0. \tag{27_2}$$

When $t = -p_{21}/p_{12}$, the first pair coincide in plane (27_2) and the second pair in plane (27_1). If we take $t = \pm p_{21} i/p_{12}$, the same situation again develops, but this time our planes of coincidence have the equations

$$(27_3) \qquad x_1 + ix_2 - x_3 - ix_4 = 0,$$

$$(27_4) \qquad x_1 - ix_2 - x_3 + ix_4 = 0.$$

From (8) we see that the two real planes of this set are on $l_{\alpha\beta}$ and the two imaginary planes are on $l_{\gamma\delta}$. It is interesting to note also that these planes are given by those values of t which give the focal points of the primary cone cubics. From the above considerations it follows that *the secondary cone cubics have four osculating planes in common, a real pair intersecting in the first cone ray and an imaginary pair intersecting in the second cone ray. Moreover in each of these planes lie two pairs of secondary cone cubic points. These pairs correspond to two of the four focal points of the primary cone cubics.* The four planes (27) may be called the *focal planes of the secondary cone cubics.*

From (17) and (1) we see that the planes π_1', π_2' are tangent to Q and the planes π_3', π_4' are tangent to Q'. The points of contact have for their coördinates

$$(28) \quad \begin{array}{lllll} (C_1'') & x_1 = p_{21}^3, & x_2 = p_{12}p_{21}^2 t, & x_3 = -p_{12}^2 p_{21} t^2, & x_4 = -p_{12}^3 t^3; \\ (C_2'') & x_1 = p_{12}^2 p_{21} t^2, & x_2 = p_{12}^3 t^3, & x_3 = -p_{21}^3, & x_4 = -p_{12}p_{21}^2 t; \\ (C_3'') & x_1 = p_{12}^2 p_{21} t^2, & x_2 = p_{12}p_{21}^2 t, & x_3 = -p_{21}^3, & x_4 = -p_{12}^3 t^3; \\ (C_4'') & x_1 = p_{21}^3, & x_2 = p_{12}^3 t^3, & x_3 = -p_{12}^2 p_{21} t^2, & x_4 = -p_{12}p_{21}^2 t. \end{array}$$

As t varies these four points trace cubics $C_1'', \cdots, C_4''$, two lying upon Q and two upon Q'. We shall call these four curves *primary contact cubics.*

The planes osculating the four curves $C_1'', \cdots, C_4''$ are given by the respective equations

$$(29) \quad \begin{array}{r} p_{12}^3 t^3 x_1 - 3p_{12}^2 p_{21} t^2 x_2 - 3p_{12}p_{21}^2 t\, x_3 + p_{21}^3 x_4 = 0, \\ 3p_{12}p_{21}^2 t\, x_1 - p_{21}^3 x_2 - p_{12}^3 t^3 x_3 + 3p_{12}^2 p_{21} t^2 x_4 = 0, \\ 3p_{12}p_{21}^2 t\, x_1 - 3p_{12}^2 p_{21} t^2 x_2 - p_{12}^3 t^3 x_3 + p_{21}^3 x_4 = 0, \\ p_{12}^3 t^3 x_1 - p_{21}^3 x_2 - 3p_{12}p_{21}^2 t\, x_3 + 3p_{12}^2 p_{21} t^2 x_4 = 0, \end{array}$$

and the points which correspond to these planes by means of L are, respectively,

$$
(30)\quad
\begin{aligned}
&(C_1''') && x_1 = 3p_{21}^2 t, && x_2 = p_{21}^2, && x_3 = p_{12}^2 t^3, && x_4 = 3p_{12}^2 t^2;\\
&(C_2''') && x_1 = p_{12}^2 t^3, && x_2 = 3p_{12}^2 t^2, && x_3 = 3p_{21}^2 t, && x_4 = p_{21}^2;\\
&(C_3''') && x_1 = p_{12}^2 t^3, && x_2 = p_{21}^2, && x_3 = 3p_{21}^2 t, && x_4 = 3p_{12}^2 t^2;\\
&(C_4''') && x_1 = 3p_{21}^2 t, && x_2 = 3p_{12}^2 t^2, && x_3 = p_{12}^2 t^3, && x_4 = p_{21}^2.
\end{aligned}
$$

Equations (30) define four new cubics $C_1''', \cdots, C_4'''$, which, by virtue of their relation to the primary contact cubics, we shall speak of as *secondary contact cubics.*

The equations of the osculating planes of the curves $C_1''', \cdots, C_4'''$ are, respectively,

$$
(31)\quad
\begin{aligned}
&p_{12}^2 t^2 x_1 - p_{12}^2 t^3 x_2 + p_{21}^2 x_3 - p_{21}^2 t\, x_4 = 0,\\
&p_{21}^2 x_1 - p_{21}^2 t\, x_2 + p_{12}^2 t^2 x_3 - p_{12}^2 t^3 x_4 = 0,\\
&p_{21}^2 x_1 - p_{12}^2 t^3 x_2 + p_{12}^2 t^2 x_3 - p_{21}^2 t\, x_4 = 0,\\
&p_{12}^2 t^2 x_1 - p_{21}^2 t\, x_2 + p_{21}^2 x_3 - p_{12}^2 t^3 x_4 - 0.
\end{aligned}
$$

A comparison of equations (31) with equations (1) and (2) shows that these planes are tangent to the complex quadrics, the locus of the points of contact being, for the first two, the cubics C_1, C_2 on Q, and for the second two the cubics C_3, C_4 on Q'. Starting with the four primary cone cubics we have thus, after four point transformations of space, returned to these same cubics, and have in the process introduced three other sets of four curves each, all of them cubics. Let us further examine this closed sequence of transformations.

To each point of space there corresponds by means of L a plane, and to this plane there corresponds by means of L_{11} a point. These two complexes thus determine a point transformation whose analytic expression we proceed to find. The point-plane correspondence determined by L* is given by

$$
(32)\quad u_1 = p_{12} x_3, \quad u_2 = -p_{21} x_4, \quad u_3 = -p_{12} x_1, \quad u_4 = p_{21} x_2.
$$

From (32) and the first of equations (23) the equations of this point transformation are easily obtained. They are

$$
\bar{x}_1 = -p_{21} x_2, \quad \bar{x}_2 = -3p_{12} x_1, \quad \bar{x}_3 = 3p_{21} x_4, \quad \bar{x}_4 = p_{12} x_3.
$$

Associating thus with L each of the complexes of (23) in turn, we obtain four such point transformations. They are

* Proj. Dif. Geom., p. 206.

$$
(33)\quad
\begin{aligned}
&(L, L_{11})\ \bar{x}_1 = -\ p_{21}x_2,\ \bar{x}_2 = -3p_{12}x_1,\ \bar{x}_3 = 3p_{21}x_4,\ \bar{x}_4 = \ p_{12}x_3;\\
&(L, L_{12})\ \bar{x}_1 = -3p_{21}x_2,\ \bar{x}_2 = -\ p_{12}x_1,\ \bar{x}_3 = \ p_{21}x_4,\ \bar{x}_4 = 3p_{12}x_3;\\
&(L, L_{13})\ \bar{x}_1 = -3p_{21}x_4,\ \bar{x}_2 = -3p_{12}x_3,\ \bar{x}_3 = \ p_{21}x_2,\ \bar{x}_4 = \ p_{12}x_1;\\
&(L, L_{14})\ \bar{x}_1 = -\ p_{21}x_4,\ \bar{x}_2 = -\ p_{12}x_3,\ \bar{x}_3 = 3p_{21}x_2,\ \bar{x}_4 = 3p_{12}x_1.
\end{aligned}
$$

We note in passing that these transformations are each of period two and hence that the point correspondences determined by them are reciprocal.

Since each quadric determines a (1,1) correspondence between the points and planes of space, we may set up a point transformation by making use of a linear complex and a quadric. For to each point there corresponds its polar plane by means of the complex and to this plane there corresponds its pole with respect to the quadric. Making use of this notion we define a second set of four point transformations, their expressions being

$$
(34)\quad
\begin{aligned}
&(L_{11}, Q)\quad \bar{x}_1 = -3x_1,\ \bar{x}_2 = -\ x_2,\ \bar{x}_3 = \ x_3,\ \bar{x}_4 = \ 3x_4;\\
&(L_{12}, Q)\quad \bar{x}_1 = -\ x_1,\ \bar{x}_2 = -3x_2,\ \bar{x}_3 = 3x_3,\ \bar{x}_4 = \ x_4;\\
&(L_{13}, Q')\quad \bar{x}_1 = -\ x_1,\ \bar{x}_2 = \ x_2,\ \bar{x}_3 = 3x_3,\ \bar{x}_4 = -3x_4;\\
&(L_{14}, Q')\quad \bar{x}_1 = -3x_1,\ \bar{x}_2 = \ 3x_2,\ \bar{x}_3 = \ x_3,\ \bar{x}_4 = -\ x_4.
\end{aligned}
$$

The eight transformations of (33) and (34), taken four at a time in the proper order, carry the four primary cone cubics through their four-phase cycle. Symbolically we have

$$
(35)\quad
\begin{aligned}
&(L_{12}, Q)\ \big[(L, L_{12})\,[(L_{11}, Q)\ [(L, L_{11})\,C_1 = C_1'] = C_1''] = C_1'''\big] = C_1,\\
&(L_{11}, Q)\ \big[(L, L_{11})\,[(L_{12}, Q)\ [(L, L_{12})\,C_2 = C_2'] = C_2''] = C_2'''\big] = C_2,\\
&(L_{14}, Q')\,\big[(L, L_{14})\,[(L_{13}, Q')\,[(L, L_{13})\,C_3 = C_3'] = C_3''] = C_3'''\big] = C_3,\\
&(L_{13}, Q')\,\big[(L, L_{13})\,[(L_{14}, Q')\,[(L, L_{14})\,C_4 = C_4'] = C_4''] = C_4'''\big] = C_4.
\end{aligned}
$$

It has been noted that the primary and secondary cone cubics determine the same set $L_{11}, L_{12}, L_{13}, L_{14}$ of linear complexes. Without difficulty it can be shown that the primary and secondary contact cubics determine the same four linear complexes, but in the order $L_{12}, L_{11}, L_{14}, L_{13}$. Many additional properties of these curves might be developed. It will be sufficient however to summarize the results of the last few paragraphs and then to state without proof a number of additional theorems whose truth can be demonstrated with the material at hand.

We find that *associated with each line element g of the general ruled surface there are sixteen projectively equivalent space cubics so related in cyclically ordered sets of four each that the points of the curves in any set lie upon the osculating planes of the curves of the preceding set.*

The curves of two of the four sets lie upon the complex quadrics associated with g, two from each set on each quadric, while the osculating planes of the curves of the other two sets are tangent to these quadrics.

The four linear complexes determined by the curves of any set are distinct, but any two sets determine the same four complexes.

Among the theorems whose proofs are left to the reader we have the following:

1. The primary (secondary) contact cubics are the loci of the poles of the osculating planes of the secondary (primary) cone cubics taken with respect to the complex quadrics, and conversely.

2. The cubics C_j^k $(j = 1, \cdots, 4;\ k = 0, \cdots, 3)$ belong to the complexes L_{1i}, C_j and C_j' belonging to L_{1i} $(i = j)$, and C_j'' and C_j''' belonging to L_{1i} where $i = j - (-1)^j$.

3. The ruled surfaces determined by the point correspondences set up between the pairs of cubics C_j^k, C_j^{k+1} $(j = 1, \cdots, 4;\ k = 0, \cdots, 3)$, by the parameter t, belong to one of the four complexes L_{1i}, those determined by C_j, C_j' and C_j', C_j'' belonging to $L_{1i}\,(i = j)$, and those determined by C_j'', C_j''' and C_j''', C_j belonging to L_{1i} where $i = j - (-1)^j$.

4. Each of the cubics C_j^k generates a surface as the line element g with which it is associated varies over the ruled surface S. Of these surfaces the eight generated by the primary cone cubics and the secondary contact cubics are projectively equivalent, as are also the eight generated by the secondary cone cubics and the primary contact cubics.

5. Four of the sixteen surfaces S_j^k generated by the cubics C_j^k are tangent to the ruled surface S, S_1' and S_1'' being tangent to S along the branch C_y of the flecnode curve, and S_1 and S_1''' tangent to S along the branch C_z of this curve. Of the remaining twelve surfaces, S_2 and S_2''' cut S along C_y, S_2' and S_2'' cut S along C_z, while S_3, S_4', S_4'', and S_3''' cut S along both C_y and C_z.

6. The point correspondence existing between each pair of primary cone cubics C_j determines a ruled surface on which this pair of cubics are directrix curves. Of the six surfaces thus determined, those given by C_1, C_2 and by C_3, C_4 are the two complex quadrics. The remaining four are cubic cones with vertices at the points P_α, P_β. Their equations, in the system of coördinates here employed, are

$$(C_1, C_3)\quad p_{12}^2 x_2 (x_1 + x_3)^2 + p_{21}^2 x_4 (x_2 + x_4)^2 = 0, \quad \text{vertex at } P_\alpha;$$

$$(C_1, C_4)\quad p_{12}^2 x_1 (x_1 + x_3)^2 + p_{21}^2 x_3 (x_2 + x_4)^2 = 0, \quad \text{"}\ \ \text{"}\ P_\beta;$$

$$(C_2, C_3)\quad p_{12}^2 x_3 (x_1 + x_3)^2 + p_{21}^2 x_1 (x_2 + x_4)^2 = 0, \quad \text{"}\ \ \text{"}\ P_\beta;$$

$$(C_2, C_4)\quad p_{12}^2 x_4 (x_1 + x_3)^2 + p_{21}^2 x_2 (x_2 + x_4)^2 = 0, \quad \text{"}\ \ \text{"}\ P_\alpha.$$

7. The primary contact cubics determine among themselves six ruled surfaces two of which are the complex quadrics, and the remaining four, cubic cones. The equations of the latter four are

$$(C_1'', C_3'') \quad x_4 (x_1 - x_3)^2 + x_2 (x_2 - x_4)^2 = 0, \quad \text{vertex at } P_\gamma;$$
$$(C_1'', C_4'') \quad x_3 (x_1 - x_3)^2 + x_1 (x_2 - x_4)^2 = 0, \quad \text{„ „ } P_\delta;$$
$$(C_2'', C_3'') \quad x_1 (x_1 - x_3)^2 + x_3 (x_2 - x_4)^2 = 0, \quad \text{„ „ } P_\delta;$$
$$(C_2'', C_4'') \quad x_2 (x_1 - x_3)^2 + x_4 (x_2 - x_4)^2 = 0, \quad \text{„ „ } P_\gamma.$$

No attempt has been made in this paper to investigate the properties of the loci which the points, lines, curves and surfaces here discussed will generate when the line-element g with which they are associated varies over the surface S. Nor has it been thought advisable to consider the results of imposing upon S any special conditions. The methods of attacking all of these problems are available and their solutions, while requiring some ingenuity, should involve no great difficulties.

UNIVERSITY OF WASHINGTON,
SEATTLE, WASH.